

# ON POSITIVITY FOR GENERALIZED CLUSTER VARIABLES OF AFFINE QUIVERS

XUEQING CHEN, MING DING AND FAN XU

**ABSTRACT.** It has been proved in [19] that cluster variables in cluster algebras of every skew-symmetric cluster algebra are positive. We prove that any regular generalized cluster variable of an affine quiver is positive. As a corollary, we obtain that generalized cluster variables of affine quivers are positive and construct various positive bases. This generalizes the results in [12] and [8].

## 1. INTRODUCTION

Cluster algebras were introduced by S. Fomin and A. Zelevinsky [14] in order to develop a combinatorial approach to study problems of total positivity and canonical bases in quantum groups. A cluster algebra  $\mathcal{A}$  is a subring of the field  $\mathbb{Q}(x_1, \dots, x_n)$  of rational fractions in  $n$  indeterminates, and defined via a set of generators constructed recursively. These generators are called cluster variables and are grouped into subsets of fixed finite cardinality called clusters. Monomials in the variables belonging to the same cluster are called cluster monomials. By the Laurent phenomenon [14], it is well-known that  $\mathcal{A} \subset \bigcap_c \mathbb{Z}[c^{\pm 1}]$  where  $c$  runs over the clusters in  $\mathcal{A}$ . An element  $x \in \mathcal{A}$  is called positive if  $x \in \bigcap_c \mathbb{N}[c^{\pm 1}]$  where  $c$  runs over the clusters in  $\mathcal{A}$ . It is conjectured that cluster variables are always positive [14]. It has been proved in [19] that the positivity conjecture holds for the class of skew-symmetric cluster algebras.

Various  $\mathbb{Z}$ -bases were constructed in the cluster algebras  $\mathcal{A}(Q)$  [24, 4, 16, 10, 8, 15]. When  $Q$  is an affine quiver, these bases can be expressed as a disjoint union of the set of cluster monomials and a set of generalized cluster variables  $X_M$  associated to some non-rigid regular  $kQ$ -modules  $M$ . Hence, studying the positivity for regular generalized cluster variables is helpful for us to construct canonical bases of cluster algebras.

For types  $\tilde{A}$  and  $\tilde{D}$ , G. Dupont has proved [12, Corollary 5.5] that if  $M$  is an indecomposable regular module in an exceptional tube  $\mathcal{R}$ , then  $X_M \in \mathcal{A}(Q) \cap \mathbb{N}[c^{\pm 1}]$  for any cluster  $c$  which is *compatible* with  $\mathcal{R}$  (see Definition 2.1).

In this paper, we focus on cluster algebras of affine quivers, that is of type  $\tilde{A}$ ,  $\tilde{D}$  or  $\tilde{E}$  and prove that the coefficients of Laurent expansions in normalized Chebyshev polynomials of the generalized cluster variable associated to quasi-simple modules in homogeneous tubes are positive integer (Proposition 3.2). As an application, we deduce the positivity in regular generalized cluster variables (Theorem 3.3) and

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obtain various positive integral bases in cluster algebras of affine quivers (Corollary 3.4).

## 2. PRELIMINARY

Let  $Q$  be an acyclic quiver with vertex set  $Q_0 = \{1, 2, \dots, n\}$  and we denote by  $\mathcal{A}(Q)$  the associated cluster algebra. Let  $\mathbb{C}$  be the complex number field and  $A = \mathbb{C}Q$  be the path algebra of  $Q$  and we denote by  $P_i$  (*resp.*  $I_i$ ) the indecomposable projective (*resp.* injective)  $\mathbb{C}Q$ -module with the simple top (*resp.* socle)  $S_i$  corresponding to  $i \in Q_0$ . A connected component in the Auslander-Reiten quiver of  $\mathbb{C}Q$ -modules is called regular if it doesn't contain any projective or injective  $\mathbb{C}Q$ -module. A  $\mathbb{C}Q$ -module is called regular if all of its indecomposable direct summands belong to regular components. Every regular component is of the form  $\mathbb{Z}\mathbb{A}_\infty/(p)$  for  $p \geq 0$ . If  $p \geq 1$ , the regular component is called a tube of rank  $p$ . If  $p > 1$ , the regular component is called an exceptional tube. If  $p = 1$ , the regular component is called a homogeneous tube. If  $p = 0$ , the regular component is called a sheet.

Let  $\mathcal{D}^b(Q)$  be the bounded derived category of  $\text{mod } \mathbb{C}Q$  with the shift functor  $T$  and the AR-translation  $\tau$ . The cluster category associated to  $Q$  was introduced in [1] in the spirit of categorification of cluster algebras. It is the orbit category  $\mathcal{C}(Q) := \mathcal{D}^b(Q)/F$  with  $F = [1] \circ \tau^{-1}$ . Let  $\mathbb{Q}(x_1, \dots, x_n)$  be a transcendental extension of  $\mathbb{Q}$ . The Caldero-Chapton map of an acyclic quiver  $Q$  is the map

$$X_Q^Q : \text{obj}(\mathcal{C}(Q)) \rightarrow \mathbb{Q}(x_1, \dots, x_n)$$

defined in [2] by the following rules:

- (1) if  $M$  is an indecomposable  $\mathbb{C}Q$ -module, then

$$X_M^Q = \sum_{\underline{e}} \chi(\text{Gr}_{\underline{e}}(M)) \prod_{i \in Q_0} x_i^{-\langle \underline{e}, s_i \rangle - \langle s_i, \underline{\dim} M - \underline{e} \rangle};$$

- (2) if  $M = P_i[1]$  is the shift of the projective module associated to  $i \in Q_0$ , then

$$X_M^Q = x_i;$$

- (3) for any two objects  $M, N$  of  $\mathcal{C}_Q$ , we have

$$X_{M \oplus N}^Q = X_M^Q X_N^Q.$$

Here, we denote by  $\langle -, - \rangle$  the Euler form on  $\mathbb{C}Q$ -modules and  $\text{Gr}_{\underline{e}}(M)$  is the  $\underline{e}$ -Grassmannian of  $M$ , i.e. the variety of submodules of  $M$  with dimension vector  $\underline{e}$ .  $\chi(\text{Gr}_{\underline{e}}(M))$  denote its Euler-Poincaré characteristic. We note that the indecomposable  $\mathbb{C}Q$ -modules and  $P_i[1]$  for  $i \in Q_0$  exhaust the indecomposable objects of the cluster category  $\mathcal{C}(Q)$ . For any object  $M \in \mathcal{C}(Q)$ ,  $X_M^Q$  will be called the generalized cluster variable for  $M$ . If  $M$  is a regular  $\mathbb{C}Q$ -module,  $X_M^Q$  will be called the regular generalized cluster variable. In [22], the author generalized the Caldero-Chapoton map for any cluster-tilting object  $T$  which is called the cluster character associated to  $T$ .

We recall that the normalized Chebyshev polynomial of the first kind is defined by:

$$\begin{aligned} F_0(x) &= 2, F_1(x) = x, \\ F_n(x) &= xF_{n-1}(x) - F_{n-2}(x) \quad \text{for any } n \geq 2. \end{aligned}$$

It is known that  $F_n$  is characterized by

$$F_n(t + t^{-1}) = t^n + t^{-n}$$

Following the terminology of [8, 10], we denote the generalized cluster variables associated to indecomposable regular modules with quasi-length  $n$  in homogeneous tubes for affine type by  $X_{n\delta}$ .

**Definition 2.1.** [12] (1) A cluster-tilting object  $T$  and a regular component  $\mathcal{R}$  are called to be compatible if  $\mathcal{R}$  does not contain any indecomposable direct summand of  $T$  as a quasi-simple module.

(2) Let  $T$  be any cluster-tilting object in  $\mathcal{C}(Q)$ . Set  $F_T = \text{Hom}_{\mathcal{C}(Q)}(T, -)$ ,  $B = \text{End}_{\mathcal{C}(Q)}(T)$  and  $c_T = \{c_i | i \in Q_0\}$ . For any object  $M$  in  $\mathcal{C}(Q)$  which is not in  $\text{add}(T[1])$  and for any  $\underline{e} \in K_0(\text{mod} - B)$ , the  $\underline{e}$ -component of  $X_M^T$  is

$$X_M^T(\underline{e}) = \chi(\text{Gr}_{\underline{e}}(F_T M)) \prod_{i \in Q_0} c_i^{\langle S_i, \underline{e} \rangle_a - \langle S_i, F_T M \rangle}$$

(see [22] for definitions of  $\langle -, - \rangle$  and  $\langle -, - \rangle_a$ ). The interior of  $X_M^T$  as

$$\text{int}(X_M^T) = X_M^T - (X_M^T(0) + X_M^T([F_T M]))$$

The following result which was proved in [12] will be useful for us to prove the positivity.

**Theorem 2.2.** [12] Let  $Q$  be a quiver of affine types with at least three vertices. Let  $T$  be a cluster-tilting object such that there exists an exceptional tube  $\mathcal{T}$  compatible with  $T$ . Assume that  $\text{int}(X_M^T) \in \mathbb{N}[c_T^{\pm 1}]$  for any quasi-simple module  $M$  in  $\mathcal{T}$ . Then, for any  $n \geq 1$ , the following holds:

$$F_n(X_\delta) \in \mathcal{A}(Q) \cap \mathbb{N}[c_T^{\pm 1}].$$

In [5], the authors has solved the positivity conjecture for all cluster variables of all skew-symmetric quantum cluster algebras. In particular, we have the following result.

**Theorem 2.3.** [5] Let  $Q$  be a quiver of affine types. Let  $T$  be any cluster-tilting object in  $\mathcal{C}(Q)$  and  $M$  be an indecomposable rigid object not in  $\text{add}(T[1])$ . Then, for any  $\underline{e} \in K_0(\text{mod} - B)$ , we have

$$\chi(\text{Gr}_{\underline{e}}(F_T M)) \geq 0.$$

According to Theorem 2.2 and Theorem 2.3, we can deduce the following corollary which are proved in [12] for types  $\tilde{A}$  and  $\tilde{D}$ :

**Corollary 2.4.** Let  $Q$  be a quiver of affine types with at least three vertices. Let  $T$  be a cluster-tilting object such that there exists an exceptional tube  $\mathcal{T}$  compatible with  $T$ . Then, for any  $n \geq 1$ , the following holds:

$$F_n(X_\delta) \in \mathcal{A}(Q) \cap \mathbb{N}[c_T^{\pm 1}].$$

### 3. POSITIVITY FOR REGULAR GENERALIZED CLUSTER VARIABLES OF AFFINE QUIVERS

In this section, we will prove the positivity for regular generalized cluster variables of affine quivers.

**Lemma 3.1.** *Let  $Q$  be a quiver of affine types and  $T$  be a cluster-tilting object of  $\mathcal{C}(Q)$  such that each exceptional regular component  $\mathcal{R}$  contains at least an indecomposable direct summand of  $T$ . Then there exists some  $i$  such that  $T_i$  is not a regular module and satisfies that  $\dim_{\mathbb{C}} \text{Ext}_{\mathcal{C}(Q)}^1(T_i, M(\delta)) = 1$ , where  $M(\delta)$  is a regular module with dimension vector  $\delta$  in any homogeneous tube.*

*Proof.* It is known that the shift functor  $[1] = \tau$  induces an equivalence of cluster categories and maps cluster-tilting object to another cluster-tilting object. For types  $\tilde{A}$ , such  $T_i$  is always exists because of  $\delta = (1, \dots, 1)$ . Now we consider the types  $\tilde{D}$  and  $\tilde{E}$ . In these cases, we can also find such  $T_i$ , since then the cluster-tilting object  $T$  contains at least an indecomposable direct summand in the exceptional regular component of rank 2.  $\square$

For any object  $M \oplus \oplus_i P_i[1]$  in the cluster category  $\mathcal{C}(Q)$  where  $M$  is a  $\mathbb{C}Q$ -module and  $P_i$  is a projective  $\mathbb{C}Q$ -module associated to the vertex  $i$ , recall that the dimension vector is defined by

$$\dim(M \oplus \oplus_i P_i[1]) = \dim M - \sum_i \dim S_i,$$

where  $S_i$  is a simple  $\mathbb{C}Q$ -module associated to the vertex  $i$ . We are now ready to prove the following result.

**Proposition 3.2.** *Let  $Q$  be a quiver of affine types, then*

$$F_n(X_\delta) \in \mathcal{A}(Q) \cap \mathbb{N}[c^{\pm 1}]$$

*for any cluster  $c$ .*

*Proof.* When  $Q$  is a Kronecker quiver, it has been proved in [24]. In the following, we always assume that the affine quiver  $Q$  with at least three vertices.

Let  $T = \bigoplus_{i=1}^n T_i$  be any cluster-tilting object in  $\mathcal{C}(Q)$ , we only need to prove that  $F_n(X_\delta) \in \mathcal{A}(Q) \cap \mathbb{N}[X_{T_1}^{\pm 1}, \dots, X_{T_n}^{\pm 1}]$ .

According to Corollary 2.4, we only need to prove it for such cluster-tilting object  $T$  satisfying that each exceptional regular component  $\mathcal{R}$  contains at least an indecomposable direct summand of  $T$ . By Lemma 3.1, there exists some  $i$  such that  $T_i$  is not a regular module and satisfies that  $\dim \text{Ext}_{\mathcal{C}(Q)}^1(T_i, M(\delta)) = 1$ . Then by the one dimension cluster multiplication formulas in [3], we get

$$X_\delta X_{T_i} = X_E + X_{E'}.$$

Remark that  $E$  and  $E'$  are both indecomposable rigid objects but not regular  $\mathbb{C}Q$ -modules. For convenience, we denote the rigid object in  $\mathcal{C}(Q)$  whose dimension vector  $\underline{e}$  by  $M(\underline{e})$ . Thus we have  $M(\underline{\dim} T_i) = T_i$ , as they are both rigid objects with the same dimension vectors.

We now prove the following claim by induction:

$$F_n(X_\delta) X_{T_i} = X_{M(\underline{\dim} T_i + n\delta)} + X_{M(\underline{\dim} T_i - n\delta)} \quad \text{for } \underline{\dim} T_i > n\delta;$$

$$F_n(X_\delta) X_{T_i} = X_{M(\underline{\dim} T_i + n\delta)} + X_{M(n\delta - \underline{\dim} T_i)[-1]} \quad \text{for } \underline{\dim} T_i < n\delta.$$

Firstly suppose that  $n = 1$ , by the above discussions, we have

$$X_\delta X_{T_i} = X_{M(\underline{\dim} T_i + \delta)} + X_{M(\underline{\dim} T_i - \delta)} \quad \text{for } \underline{\dim} T_i > \delta;$$

$$X_\delta X_{T_i} = X_{M(\underline{\dim} T_i + \delta)} + X_{M(\delta - \underline{\dim} T_i)[-1]} \quad \text{for } \underline{\dim} T_i < \delta.$$

Secondly suppose that  $n = 2$ , which need to be divided into the following two cases:

(1) If  $\underline{\dim} T_i > \delta$ , we have

$$\begin{aligned} F_2(X_\delta)X_{T_i} &= (X_\delta^2 - 2)X_{T_i} \\ &= X_\delta(X_{M(\underline{\dim} T_i + \delta)} + X_{M(\underline{\dim} T_i - \delta)}) - 2X_{T_i} \\ &= X_{M(\underline{\dim} T_i + 2\delta)} + X_{T_i} + X_\delta X_{M(\underline{\dim} T_i - \delta)} - 2X_{T_i}. \end{aligned}$$

We compute  $X_\delta X_{M(\underline{\dim} T_i - \delta)}$ , which can be solved in the following cases:

In the case that  $\underline{\dim} T_i > 2\delta$ , we have

$$X_\delta X_{M(\underline{\dim} T_i - \delta)} = X_{T_i} + X_{M(\underline{\dim} T_i - 2\delta)},$$

thus

$$F_2(X_\delta)X_{T_i} = X_{M(\underline{\dim} T_i + 2\delta)} + X_{M(\underline{\dim} T_i - 2\delta)};$$

In the case that  $\underline{\dim} T_i < 2\delta$ , we have

$$X_\delta X_{M(\underline{\dim} T_i - \delta)} = X_{T_i} + X_{M(2\delta - \underline{\dim} T_i)[-1]},$$

thus

$$F_2(X_\delta)X_{T_i} = X_{M(\underline{\dim} T_i + 2\delta)} + X_{M(2\delta - \underline{\dim} T_i)[-1]}.$$

(2) If  $\underline{\dim} T_i < \delta$ , we have

$$\begin{aligned} F_2(X_\delta)X_{T_i} &= (X_\delta^2 - 2)X_{T_i} \\ &= X_\delta(X_{M(\underline{\dim} T_i + \delta)} + X_{M(\delta - \underline{\dim} T_i)[-1]}) - 2X_{T_i} \\ &= X_{M(\underline{\dim} T_i + 2\delta)} + X_{T_i} + X_{M(2\delta - \underline{\dim} T_i)[-1]} + X_{T_i} - 2X_{T_i} \\ &= X_{M(\underline{\dim} T_i + 2\delta)} + X_{M(2\delta - \underline{\dim} T_i)[-1]}. \end{aligned}$$

Now suppose that the above equations hold for all  $k \leq n$ , we need to prove them for  $k = n + 1$ , which can be divided into the following two cases:

(1) If  $\underline{\dim} T_i > n\delta$ , we have

$$\begin{aligned} F_{n+1}(X_\delta)X_{T_i} &= (X_\delta F_n(X_\delta) - F_{n-1}(X_\delta))X_{T_i} \\ &= X_\delta(X_{M(\underline{\dim} T_i + n\delta)} + X_{M(\underline{\dim} T_i - n\delta)}) \\ &\quad - (X_{M(\underline{\dim} T_i + (n-1)\delta)} + X_{M(\underline{\dim} T_i - (n-1)\delta)}) \\ &= X_{M(\underline{\dim} T_i + (n+1)\delta)} + X_{M(\underline{\dim} T_i + (n-1)\delta)} + X_\delta X_{M(\underline{\dim} T_i - n\delta)} \\ &\quad - (X_{M(\underline{\dim} T_i + (n-1)\delta)} + X_{M(\underline{\dim} T_i - (n-1)\delta)}). \end{aligned}$$

We need to compute  $X_\delta X_{M(\underline{\dim} T_i - n\delta)}$ :

In the case that  $\underline{\dim} T_i > (n+1)\delta$ , we have

$$X_\delta X_{M(\underline{\dim} T_i - n\delta)} = X_{M(\underline{\dim} T_i - (n-1)\delta)} + X_{M(\underline{\dim} T_i - (n+1)\delta)},$$

Thus we have

$$F_{n+1}(X_\delta)X_{T_i} = X_{M(\underline{\dim} T_i + (n+1)\delta)} + X_{M(\underline{\dim} T_i - (n+1)\delta)};$$

In the case that  $\underline{\dim} T_i < (n+1)\delta$ , we have

$$X_\delta X_{M(\underline{\dim} T_i - n\delta)} = X_{M(\underline{\dim} T_i - (n-1)\delta)} + X_{M((n+1)\delta - \underline{\dim} T_i)[-1]},$$

thus

$$F_{n+1}(X_\delta)X_{T_i} = X_{M(\underline{\dim} T_i + (n+1)\delta)} + X_{M((n+1)\delta - \underline{\dim} T_i)[-1]}.$$

(2) If  $\underline{\dim} T_i < n\delta$ , we have

$$\begin{aligned}
F_{n+1}(X_\delta)X_{T_i} &= (X_\delta F_n(X_\delta) - F_{n-1}(X_\delta))X_{T_i} \\
&= X_\delta(X_{M(\underline{\dim}T_i+n\delta)} + X_{M(n\delta-\underline{\dim}T_i)[-1]}) - F_{n-1}(X_\delta)X_{T_i} \\
&= X_{M(\underline{\dim}T_i+(n+1)\delta)} + X_{M(\underline{\dim}T_i+(n-1)\delta)} \\
&\quad + X_\delta X_{M(n\delta-\underline{\dim}T_i)[-1]} - F_{n-1}(X_\delta)X_{T_i}.
\end{aligned}$$

When  $\underline{\dim}T_i > (n-1)\delta$ , we have

$$X_\delta X_{M(n\delta-\underline{\dim}T_i)[-1]} = X_{M((n+1)\delta-\underline{\dim}T_i)[-1]} + X_{M(\underline{\dim}T_i-(n-1)\delta)},$$

and

$$F_{n-1}(X_\delta)X_{T_i} = X_{M(\underline{\dim}T_i+(n-1)\delta)} + X_{M(\underline{\dim}T_i-(n-1)\delta)},$$

then we have

$$F_{n+1}(X_\delta)X_{T_i} = X_{M(\underline{\dim}T_i+(n+1)\delta)} + X_{M((n+1)\delta-\underline{\dim}T_i)[-1]};$$

When  $\underline{\dim}T_i < (n-1)\delta$ , we obtain

$$X_\delta X_{M(n\delta-\underline{\dim}T_i)[-1]} = X_{M((n+1)\delta-\underline{\dim}T_i)[-1]} + X_{M((n-1)\delta-\underline{\dim}T_i)[-1]},$$

and

$$F_{n-1}(X_\delta)X_{T_i} = X_{M(\underline{\dim}T_i+(n-1)\delta)} + X_{M((n-1)\delta-\underline{\dim}T_i)[-1]},$$

thus we have

$$F_{n+1}(X_\delta)X_{T_i} = X_{M(\underline{\dim}T_i+(n+1)\delta)} + X_{M((n+1)\delta-\underline{\dim}T_i)[-1]}.$$

The claim is proved.

Note that  $X_{M(\underline{\dim}T_i+n\delta)}$ ,  $X_{M(\underline{\dim}T_i-n\delta)}$  and  $X_{M(n\delta-\underline{\dim}T_i)[-1]}$  are all cluster variables, then by [19], they belong to  $\mathcal{A}(Q) \cap \mathbb{N}[X_{T_1}^{\pm 1}, \dots, X_{T_n}^{\pm 1}]$ . Thus, by the above proved claim, we obtain

$$F_n(X_\delta) \in \mathcal{A}(Q) \cap \mathbb{N}[X_{T_1}^{\pm 1}, \dots, X_{T_n}^{\pm 1}].$$

Therefore the result follows.  $\square$

Now we make use of Proposition 3.2 to prove the main result in this paper.

**Theorem 3.3.** *Let  $Q$  be a quiver of affine types, then*

$$X_M \in \mathcal{A}(Q) \cap \mathbb{N}[c^{\pm 1}]$$

*for any object  $M$  in  $\mathcal{C}(Q)$  and any cluster  $c$ .*

*Proof.* For any object  $M, N \in \mathcal{C}(Q)$ , we have  $X_{M \oplus N} = X_M X_N$ , and also it is well-known that cluster variables are positive, so we only need to prove the theorem for any indecomposable regular generalized variables.

Firstly, we consider the case in homogeneous tubes. Note that  $X_{n\delta} = F_n(X_\delta) + X_{(n-2)\delta}$ , thus we can prove  $X_{n\delta} \in \mathbb{N}[c^{\pm 1}]$  by induction.

Secondly, we consider the case in non-homogeneous tubes. We fix a non-homogeneous tube  $\mathcal{T}$  of rank  $r$ . The quasi-simples of  $\mathcal{T}$  are denoted by  $E_i$  with  $1 \leq i \leq r$  ordered so that  $\tau E_i = E_{i-1}$ . The regular module with quasi-socle  $E$  and quasi-length  $k$  for any  $k \in \mathbb{N}$  are denoted by  $E[k]$ . According to the general different property (see [11, Theorem 3.4]):

$$X_{E_i[nr+k]} = X_{E_i[k]} F_n(X_\delta) + X_{E_{i+k+1}[nr-k-2]}$$

where  $r$  is the rank of an exceptional tube,  $n \geq 0$  and  $0 \leq k \leq r-1$ , and the positivity of  $F_n(X_\delta)$  which is proved in Proposition 3.2, we can deduce  $X_{E_i[nr+k]} \in$

$\mathcal{A}(Q) \cap \mathbb{N}[c^{\pm 1}]$  by induction. Here we only need to note that  $X_\delta X_{E_i[r-1]} = X_{E_i[2r-1]}$  where  $0 \leq i \leq r-1$ .  $\square$

Let  $Q$  be a quiver of affine types, then we have the following three integral bases of the cluster algebras  $\mathcal{A}(Q)$  (see [8, 10, 11]):

$$\mathcal{B} = \mathcal{CM} \cup \{F_n(X_\delta)X_R | R \text{ is a regular rigid } kQ\text{-module and } n \geq 1\}$$

$$\mathcal{S} = \mathcal{CM} \cup \{X_{n\delta}X_R | R \text{ is a regular rigid } kQ\text{-module and } n \geq 1\}$$

$$\mathcal{G} = \mathcal{CM} \cup \{X_\delta^n X_R | R \text{ is a regular rigid } kQ\text{-module and } n \geq 1\}$$

where we denote the set of all cluster monomials of the cluster algebras  $\mathcal{A}(Q)$  by  $\mathcal{CM}$ .

We can now put all these results together to obtain:

**Corollary 3.4.** *Let  $Q$  be a quiver of affine types and  $c$  be any cluster, then we have*

- (1)  $\mathcal{B} \in \mathcal{A}(Q) \cap \mathbb{N}[c^{\pm 1}]$ ;
- (2)  $\mathcal{S} \in \mathcal{A}(Q) \cap \mathbb{N}[c^{\pm 1}]$ ;
- (3)  $\mathcal{G} \in \mathcal{A}(Q) \cap \mathbb{N}[c^{\pm 1}]$ ;

*Proof.* By Proposition 3.2 and the positivity in cluster variables, we obtain that

$$\mathcal{B} \in \mathcal{A}(Q) \cap \mathbb{N}[c^{\pm 1}].$$

By Theorem 3.3 and the positivity in cluster variables, we obtain that

$$\mathcal{S} \in \mathcal{A}(Q) \cap \mathbb{N}[c^{\pm 1}] \text{ and } \mathcal{G} \in \mathcal{A}(Q) \cap \mathbb{N}[c^{\pm 1}].$$

The result follows.  $\square$

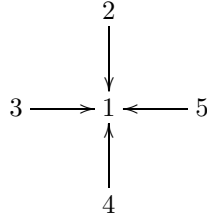
**Remark 3.5.** (1) The basis  $\mathcal{B}$  was initially constructed for rank 2 cluster algebras of finite and affine types in [24] and for type  $\tilde{A}_2^{(1)}$  in [15] where are called canonical bases, and then constructed for types  $A$  and  $\tilde{A}$  in [13] where are called the atomic basis.

(2) The basis  $\mathcal{B}$  was constructed for the Kronecker quiver [4] where is called the dual semicanonical basis.

(3) The basis  $\mathcal{G}$  was constructed in [10] for type  $\tilde{A}$  and [8] for affine types, and for more general case in [17] where are called the generic basis.

#### 4. AN EXAMPLE: TYPE $\tilde{D}_4$

We consider the tame quiver  $Q$  of type  $\tilde{D}_4$  as follows



In this case, we will provide an explicit description of the proof in Proposition 3.2.

The category of regular modules decomposes into a direct sum of tubes indexed by the projective line  $\mathbb{P}^1$  among which there are just three tubes of rank 2 and all other tubes are homogeneous tubes [6]. We denote these three exceptional

tubes labelled by the subset  $\{0, 1, \infty\}$  of  $\mathbb{P}^1$ . The quasi-simple modules in non-homogeneous tubes are denoted by  $E_1, E_2, E_3, E_4, E_5, E_6$ , where

$$\underline{\dim}(E_1) = (1, 1, 1, 0, 0), \underline{\dim}(E_2) = (1, 0, 0, 1, 1), \underline{\dim}(E_3) = (1, 1, 0, 1, 0),$$

$$\underline{\dim}(E_4) = (1, 0, 1, 0, 1), \underline{\dim}(E_5) = (1, 0, 1, 1, 0), \underline{\dim}(E_6) = (1, 1, 0, 0, 1).$$

We remark that  $\{E_1, E_2\}$ ,  $\{E_3, E_4\}$  and  $\{E_5, E_6\}$  are pairs of the quasi-simple modules at the mouth of exceptional tubes labelled by  $1, \infty$  and  $0$ , respectively. We denote the minimal imaginary root by  $\delta = (2, 1, 1, 1, 1)$ . From the Auslander-Reiten quiver of type  $\tilde{D}_4$ , we know that all preprojective modules are the following forms:

- 1):  $C(n)$  with  $\underline{\dim}C(n) = (2n - 1, n - 1, n - 1, n - 1, n - 1)$ , where  $n \geq 1$ .
- 2):  $M_1(n), M_2(n), M_3(n), M_4(n)$  with

$$\underline{\dim}M_1(n) = (n, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}),$$

$$\underline{\dim}M_2(n) = (n, \frac{n-1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2}),$$

$$\underline{\dim}M_3(n) = (n, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n+1}{2}, \frac{n-1}{2}),$$

$$\underline{\dim}M_4(n) = (n, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n+1}{2}),$$

where  $n$  is odd and  $n \geq 1$ .

- 3):  $N_1(n), N_2(n), N_3(n), N_4(n)$  with :

$$\underline{\dim}N_1(n) = (n, \frac{n-2}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}),$$

$$\underline{\dim}N_2(n) = (n, \frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}, \frac{n}{2}),$$

$$\underline{\dim}N_3(n) = (n, \frac{n}{2}, \frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}),$$

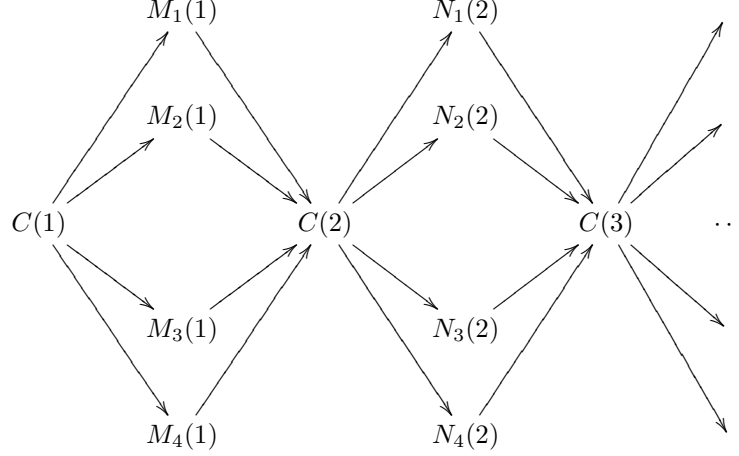
$$\underline{\dim}N_4(n) = (n, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n-2}{2}),$$

where  $n$  is even and  $n \geq 2$ . Note that

$$C(1) = P_1, M_1(1) = P_2, M_2(1) = P_3, M_3(1) = P_4, M_4(1) = P_5.$$



The Auslander-Reiten quiver of the preprojective component is as follows:



All preinjective modules are the following forms:

1):  $C'(n)$  with  $\underline{\dim} C'(n) = (2n-1, n, n, n, n)$ , where  $n \geq 1$ .

2):  $M'_1(n), M'_2(n), M'_3(n), M'_4(n)$  with When

$$\underline{\dim} M'_1(n) = (n-1, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}),$$

$$\underline{\dim} M'_2(n) = (n-1, \frac{n-1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2}),$$

$$\underline{\dim} M'_3(n) = (n-1, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n+1}{2}, \frac{n-1}{2}),$$

$$\underline{\dim} M'_4(n) = (n-1, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n+1}{2}),$$

where  $n$  is odd and  $n \geq 1$ .

3):  $N'_1(n), N'_2(n), N'_3(n), N'_4(n)$  with

$$\underline{\dim} N'_1(n) = (n-1, \frac{n-2}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}),$$

$$\underline{\dim} N'_2(n) = (n-1, \frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}, \frac{n}{2}),$$

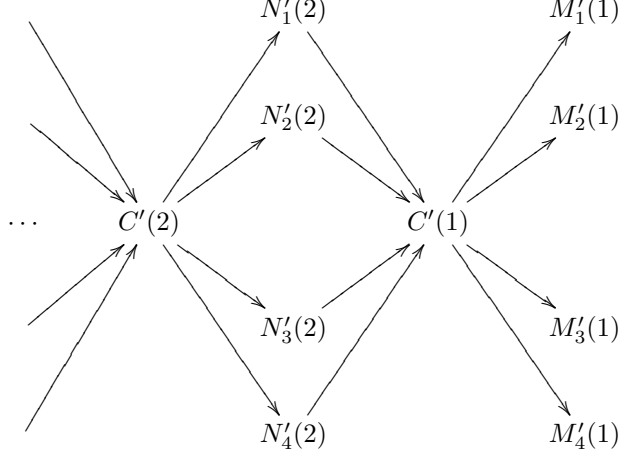
$$\underline{\dim} N'_3(n) = (n-1, \frac{n}{2}, \frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}),$$

$$\underline{\dim} N'_4(n) = (n-1, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n-2}{2}),$$

where  $n$  is even and  $n \geq 2$ . Note that

$$C'(1) = I_1, M'_1(1) = I_2, M'_2(1) = I_3, M'_3(1) = I_4, M'_4(1) = I_5.$$

The Auslander-Reiten quiver of the preinjective component is as follows:



The following result proved in [7] is useful for us to get the positivity.

**Proposition 4.1.** [7] *Assume that  $n \geq 1$ , then we have*

(1) *If  $n = 1$ , then*

$$X_\delta X_{P_2} = X_{M_1(3)} + X_{M'_1(1)};$$

(2) *If  $n \geq 3$  is odd, then*

$$X_\delta X_{M_1(n)} = X_{M_1(n+2)} + X_{M_1(n-2)}.$$

Similar to Proposition 4.1(2), it is easy to show the following result directly.

**Proposition 4.2.** *If  $n \geq 3$  is odd, then*

$$X_\delta X_{M'_1(n)} = X_{M'_1(n+2)} + X_{M'_1(n-2)}.$$

For any cluster-tilting object  $T$ , it is easy to see that there exists at least a direct summand  $T_i$  of  $T$  such that  $\tau^m T_i$  is equal to some  $P_k$  for  $2 \leq k \leq 5$  and some certain  $m \in \mathbb{Z}$ . Note that  $\tau$  induces an equivalence of cluster categories and maps cluster-tilting object to another cluster-tilting object. Without loss of generality, we can assume that  $T_i = P_2$ , then we have the following result.

**Proposition 4.3.** *Assume that  $n \geq 1$ , then we have*

$$F_n(X_\delta)X_{P_2} = X_{M_1(2n+1)} + X_{M'_1(2n-1)}.$$

*Proof.* We prove it by induction. When  $n = 1$ , it follows from the Proposition 4.1(1). When  $n = 2$ , note that  $P_2 = M_1(1)$ , we have

$$\begin{aligned} F_2(X_\delta)X_{P_2} &= (X_\delta^2 - 2)X_{P_2} \\ &= X_\delta(X_{M_1(3)} + X_{M'_1(1)}) - 2X_{P_2} \\ &= X_{M_1(5)} + X_{M_1(1)} + X_{M'_1(3)} + X_{M_1(1)} - 2X_{P_2} \\ &= X_{M_1(5)} + X_{M'_1(3)}. \end{aligned}$$

Now suppose that it holds for  $n \leq k$ . When  $n = k + 1$ , then by Proposition 4.1 and Proposition 4.2, we have

$$\begin{aligned}
 F_{k+1}(X_\delta)X_{P_2} &= (X_\delta F_k(X_\delta) - F_{k-1}(X_\delta))X_{P_2} \\
 &= X_\delta(X_{M_1(2k+1)} + X_{M'_1(2k-1)}) - (X_{M_1(2k-1)} + X_{M'_1(2k-3)}) \\
 &= X_{M_1(2k+3)} + X_{M_1(2k-1)} + X_{M'_1(2k+1)} + X_{M'_1(2k-3)} \\
 &\quad - (X_{M_1(2k-1)} + X_{M'_1(2k-3)}) \\
 &= X_{M_1(2k+3)} + X_{M'_1(2k+1)}.
 \end{aligned}$$

Thus the proof is finished.  $\square$

It is obvious to see that  $X_{M_1(2n+1)}$  and  $X_{M'_1(2n-1)}$  are all cluster variables, so both of them belong to  $\mathcal{A}(Q) \cap \mathbb{N}[X_{T_1}^{\pm 1}, \dots, X_{T_n}^{\pm 1}]$ . Note that  $X_{P_2} = X_{T_i}$ , then by Proposition 4.3, we get

$$F_n(X_\delta) \in \mathcal{A}(Q) \cap \mathbb{N}[X_{T_1}^{\pm 1}, \dots, X_{T_n}^{\pm 1}].$$

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